

THE BLACK-AND-WHITE COLORING PROBLEM ON PERMUTATION GRAPHS

Ton Kloks

Department of Computer Science
National Tsing Hua University
Taiwan

Abstract. Given a graph G and integers b and w . The black-and-white coloring problem asks if there exist disjoint sets of vertices B and W with $|B| = b$ and $|W| = w$ such that no vertex in B is adjacent to any vertex in W . In this paper we show that the problem is polynomial when restricted to permutation graphs.

1 Introduction

Definition 1. Let $G = (V, E)$ be a graph and let b and w be two integers. A black-and-white coloring of G colors b vertices black and w vertices white such that no black vertex is adjacent to any white vertex.

In other words, the black-and-white coloring problem asks for a complete bipartite subgraph M in the complement \bar{G} of G with b and w vertices in the two color classes of M .

The black-and-white coloring problem is NP-complete for graphs in general [4]. That paper also shows that the problem can be solved for trees in $O(n^3)$ time. In a recent paper [2] the worst-case timebound for an algorithm on trees was improved to $O(n^2 \log^3 n)$ time [2]. The paper [2] mentions, among other things, a manuscript by Kobler, *et al.*, which shows that the problem can be solved in polynomial time for graphs of bounded treewidth.

In this paper we investigate the complexity of the problem for permutation graphs. An intersection model for permutation graphs is obtained as follows. Consider two horizontal lines L_1 and L_2 , one above the other. Label n distinct points on L_1 and on L_2 with labels $\{1, \dots, n\}$. For each $k \in \{1, \dots, n\}$ connect the point with label k on L_1 with the point with label k on L_2 by a straight line segment. This is called a permutation diagram. The corresponding permutation graph with vertices $\{1, \dots, n\}$ is the intersection graph of the line segments.

Permutation graphs can be recognized in linear time [7]. A permutation diagram can be obtained in linear time.

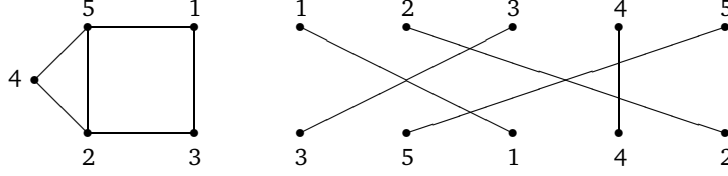


Fig. 1. A permutation graph and its permutation diagram

2 Black-and-white colorings of permutation graphs

Definition 2. Consider a permutation diagram. A scanline is a linesegment that connects a point on L_1 with a point on L_2 , such that the endpoints do not coincide with the endpoints of any line segment.

A black-and-white coloring with b black vertices and w white vertices is optimal if every uncolored vertex has a black and a white neighbor.

Lemma 1. Assume there exists an optimal black-and-white coloring of G with b black vertices and w white vertices. There exists a collection of pairwise non-intersecting scanlines such that the vertices which are uncolored are exactly the line segments that cross one or more scanlines.

Proof. Remove the line segments from the diagram of the vertices that are not colored. Each of the remaining components is colored black or white. Notice that the components form a consecutive sequence in the diagram. Place a scanline between any two consecutive components. The vertices that are not colored are precisely those that cross one of the scanlines. \square

Theorem 1. There exists a polynomial-time algorithm which checks if a permutation graph can be colored with b black and w white vertices.

Proof. Consider a permutation diagram for a permutation graph $G = (V, E)$. A piece consists of a pair of non-intersecting scanlines.

Consider the subgraph of G induced by the line segments with both endpoints between the two scanlines. Using dynamic programming, the algorithm checks if there is a black and white coloring of the piece with b' black and w' white vertices, for all values b' and w' . We describe the procedure below.

The smallest pieces consist of two scanlines such that there is exactly one line segment between them. The subgraph induced by this piece has one vertex. There are two possible optimal colorings; either the vertex is black or it is white.

Consider an arbitrary piece, say that it is bordered by scanlines s_1 and s_2 . Two possible colorings color the piece completely black or completely white. Cut the piece in two by a scanline s which is between s_1 and s_2 . Let S be the set of line segments that cross s . The vertices of s are uncolored. If there are colorings with b_1 and w_1 vertices colored black and white in the left piece and with b_2 and w_2 black and white vertices in the right piece, then the piece can be colored with $b_1 + b_2$ black vertices and $w_1 + w_2$ white vertices.

There are $O(n^4)$ different pieces, namely, there are $O(n^2)$ scanlines, and each piece is bordered by two of them. To process a piece, we try $O(n^2)$ scanlines that are between the two bordering scanlines. By table look-up, the algorithm checks if there is a black-and-white coloring of the piece with b' black vertices and w' white vertices. Each table consists of $O(n^2)$ entries. A table for the piece can be computed in $O(n^8)$ time. \square

Remark 1. A similar algorithm can be obtained for the classes of circle graphs and d-trapezoid graphs [3,5].

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